# Ensemble Learning for Reectometry

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# Ensemble Learning for Reflectometry

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**Abstract.** In "Blind Reflectometry" (Romeiro and Zickler, 2010 [3]) we describe a variational Bayesian approach to inferring material properties (BRDF) from a single image of a known shape under unknown, real-world illumination. This technical report provides additional details of that approach. First, we detail the prior probability distribution for natural lighting environments. Second, we provide a derivation of the bilinear likelihood expression that is based on discretizing the rendering equation. Third and finally, we provide the update equations for the iterative algorithm that computes an approximation to the posterior distribution of BRDFs.

## 1 Illumination Representation Details

As per Eq. 5 in [3], the prior probability distribution on the wavelet coefficients of real-world illumination environments is of the form

$$p(\ell) = \prod_{m=2}^{M} \sum_{n=1}^{N_m} \pi_{nm}\ p_{nm}(\ell_m), \tag{1}$$

with $N_m$ the number of mixture components for coefficient $m$, and $\pi_{nm}$ the mixing weights. All coefficients in any one group (defined in Sect. 3.1 [3]) share the same $N_m$, $\pi_{nm}$ and $p_{nm}$. The exact forms of the mixture components are as follows:

$$p_{nm} \sim \begin{cases} N(0, v_{nm}), \text{if } m \neq 3 \\ N_{RC}(\bar{\ell}_{nm}, v_{nm}, T), \text{if } m = 3 \end{cases}, \tag{2}$$

$$v_{nm} = \begin{cases} v_{nr(m)g(m)},\ \text{if } r(m) \in \{2,3\} \\ v_{nr(m)},\ \text{if } r(m) \in \{4,5\} \\ v_{nm},\ \text{otherwise} \end{cases} \quad \text{and } N_m = \begin{cases} r(m),\ \text{if } r(m) \in \{3,4,5\} \\ 2,\ \text{if } r(m) \in \{1,2\}, m \neq 3 \\ 1,\ \text{if } m = 3 \end{cases}$$

where $m = 3$ is the wavelet coefficient corresponding to the vertical basis element at the coarsest scale, $r(m)$ indicates the scale of coefficient $m$, and $g(m)$ indicates to which grouping (spatial location and basis element type) coefficient $m$ belongs to (see Fig. 2 in [3]). $N_{RC}$ corresponds to a Gaussian distribution rectified at a given point.

Letting the rectification point be $T$, say, the rectified Gaussian distribution has the form

$$x \sim N_{RC}(u,w,T) \leftrightarrow p(x) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi w}\mathrm{erfc}(-\frac{(u-T)}{\sqrt{2w}})} \exp\frac{-(x-u)^2}{2w}, & \text{if } x \geq T \\ 0, & \text{otherwise} \end{cases}, \quad (3)$$

and it is related to the "standard" rectified gaussian distribution by

$$x \sim N_R(u,w) \leftrightarrow x \sim N_{RC}(u,w,0). \quad (4)$$

Furthermore, if $x \sim N_{RC}(u,w,T)$, then

$$\langle x \rangle_{p(x)} = u + \frac{\sqrt{2w}}{\sqrt{\pi}\mathrm{erfcx}(-\frac{(u-T)}{\sqrt{2w}})}$$

$$\langle x^2 \rangle_{p(x)} = u\,\langle x \rangle + w + T(\langle x \rangle - u),$$

where $\langle \cdot \rangle_{p(\cdot)}$ represents the expectation under distribution $p(\cdot)$.

## 2 Discrete rendering equation

We define the following imaging model (Eq. 7 in [3] ):

$$I_i = \gamma \int_{\Omega} L_i(\omega)V_i(\omega)F_i(\omega)(n_i \cdot \omega)d\omega + \epsilon, \quad (5)$$

with $\epsilon, \gamma, L_i, V_i, F_i$ being the noise, exposure, local lighting, local visibility, and reflectance, respectively.

Given linear lighting and reflectance representations, $L = \sum \ell_m \psi_m$ and $F = \sum f_k \phi_k$, we can write $L_i(\omega) \approx \sum (R_i \ell)_r \rho_r(\omega)$ and $F_i(\omega) \approx \sum (W_i f)_r \rho_r(\omega)$, where $\{\rho_r(\omega)\}$ is the delta basis in a discretized local hemisphere $\Omega_D$ (a discretization of $\Omega$ in Eq. 5), $R_i$ is the linear transformation that maps from the wavelet basis in domain of the octahedral map into the delta basis in $\Omega_D$ (see [4]), and $W_i$ is the linear transformation that maps from the NMF basis into the delta basis in $\Omega_D$. We also represent the visibility in the delta basis in the discrete hemisphere ($V_i(\omega) \approx \sum (v_i)_r \rho_r(\omega)$), and substitute these into the rendering equation:

$$\int_{\Omega} L_i(\omega)V_i(\omega)F_i(\omega)(n_i \cdot \omega)d\omega \approx \sum_{r_1,r_2,r_3} (R_i\ell)_{r_1}(W_i f)_{r_2}(v_i)_{r_3} C_{r_1,r_2,r_3}, \quad (6)$$

$$\begin{aligned} C_{r_1,r_2,r_3} &= \int_{\Omega_D} \rho_{r_1}(\omega)\rho_{r_2}(\omega)\rho_{r_3}(\omega)(n_i \cdot \omega)d\omega \\ &= \begin{cases} \int_{\Omega_{D\rho_{r_1}}} (n_i \cdot \omega)d\omega \text{ if } r_1 = r_2 = r_3 \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

where $\Omega_{D\rho_{r_1}}$ is the support of $\rho_{r_1}(\omega)$. Thus,

$$\int_{\Omega} L_i(\omega)V_i(\omega)F_i(\omega)(n_i \cdot \omega)d\omega \approx \sum_r (R_i\ell)_r (W_i f)_r (v_i)_r C_r, \tag{7}$$

which can be written in matrix form as $\ell^T M_i f$, where the per-pixel matrices $M_i$ are given by $M_i = R_i^T \mathrm{diag}(C \circ v_i) W_i$, where $\circ$ is the Hadamard (or entrywise) product, and $\mathrm{diag}(\cdot)$ is a square matrix with its argument along the diagonal. Substituting this expression into Eq. 5, we have

$$I_i = \gamma \ell^T M_i f + \epsilon, \tag{8}$$

which is exactly Eq. 8 in [3].

## 3 Update equations for the posterior approximation

As mentioned in Sect. 3.4 of [3], the ensemble of distributions $q(\cdot)$ that approximate the posterior distribution of reflectance $f$, lighting $\ell$, and exposure and noise parameters $\gamma$, $\sigma$ are of the forms

$$q(f) = \prod q_k(f_k), \text{ with } q_k \sim N_R(u_k, w_k), \tag{9}$$

$$q(\ell) = \prod q_m(\ell_m), \text{ with } q_m \sim \begin{cases} N(u_m, w_m) & \text{if } m \neq 3 \\ N_{RC}(u_m, w_m, T) & \text{otherwise,} \end{cases} \tag{10}$$

$$q(\gamma) \sim N_R(u_\gamma; w_\gamma), \tag{11}$$

$$q(\sigma^{-2}) \sim \Gamma(\sigma^{-2}; a_p, b_p), \tag{12}$$

where $N_R$ corresponds to a Gaussian distribution rectified at 0 and $N_{RC}$ corresponds to a Gaussian distribution rectified at $T$ (see Eq. 3 above). The closed-form expressions for the updated parameters of each approximating distribution in terms of the current parameters of the others (Algorithm 1 in [3]) are as follows. In these expressions, the notation $M_{i_{mk}}$ refers to the $mk^{th}$ element of matrix $M_i$.

$$\frac{u_k}{w_k} = -\left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} C_{ki} - \lambda_k, \tag{13}$$

$$\frac{1}{w_k} = \left\langle \sigma^{-2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle (\sum_m \ell_m M_{i_{mk}})^2 \right\rangle_{q(\ell)} \left\langle \gamma^2 \right\rangle_{q(\gamma)}, \tag{14}$$

$$\frac{u_\gamma}{w_\gamma} = -\left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle -2 I_i \ell^T M_i f \right\rangle_{q(\ell,f)} - \lambda_\gamma, \tag{15}$$

$$\frac{1}{w_\gamma} = \left\langle \sigma^{-2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle (\ell^T M_i f)^2 \right\rangle_{q(\ell,f)}, \tag{16}$$

$$\frac{u_m}{w_m} = -\left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} E_{mi} + \sum_{n=1}^{N_m} \frac{\delta_{nm}\bar{\ell}_{nm}}{v_{nm}}, \tag{17}$$

$$\frac{1}{w_m} = < \sigma^{-2} >_{q(\sigma^{-2})} \sum_{i=1}^{N} < (\sum_k M_{i_{mk}} f_k)^2 >_{q(f)} < \gamma^2 >_{q(\gamma)} + \sum_{n=1}^{N_m} \frac{\delta_{nm}}{v_{nm}}, \tag{18}$$

with,

$$a_p = a + \frac{\sum_{i=1}^{N} < (\gamma \ell^T M_i f - I_i)^2 >_{q(\ell,f)}}{2}, \tag{19}$$

$$b_p = b + \frac{N}{2}, \tag{20}$$

$$C_{ki} = 2 \left\langle \left( \sum_m \sum_{s \neq k} \gamma \ell_m M_{i_{ms}} f_s - I_i \right) \left( \sum_m \gamma \ell_m M_{i_{mk}} \right) \right\rangle_{q(\ell, f_{-k}, \gamma)}, \tag{21}$$

$$E_{mi} = 2 \left\langle \left( \sum_{r \neq m} \sum_k \gamma \ell_r M_{i_{rk}} f_k - I_i \right) \left( \sum_k \gamma M_{i_{mk}} f_k \right) \right\rangle_{q(\ell_{-m}, f, \gamma)}, \tag{22}$$

$$\delta_{nm} = \alpha_m \exp\left( \langle \log\left(\pi_{nm} p_{nm}(\ell_m)\right) \rangle_{q_m(\ell_m)} \right), \tag{23}$$

where $\alpha_m$ are such that $\sum_{n=1}^{N_m} \delta_{nm} = 1$, and the notation $f_{-k}$ corresponds to the vector $f$ with its $k^{th}$ entry removed.

Simplified expressions for some of these expectations and formulas are in Sect. 5 below. But first, in order to provide some examples, the next section includes derivations of the parametric forms of two of the ensemble distributions ($q_k(f_k)$ and $q(\gamma)$) as well as the update equations for their respective parameters ($u_k, w_k$ and $u_\gamma, w_\gamma$) listed above.

### 3.1 Parametric form and update equations for $q_k(f_k)$

Following Miskin [1] and Miskin and MacKay [2], we find the form of the ensemble distribution $q_k(\cdot)$, by taking the variational derivative of $C_{KL}$ with respect to $q_k$,

$$\frac{dC_{KL}}{dq_k(f_k)} = \log q_k(f_k) - \log p_k(f_k) \tag{24}$$

$$+ \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle (\gamma \ell^T M_i f - I_i)^2 \right\rangle_{q(\ell, f_{-k}, \gamma)} \tag{25}$$

$$+ 1 + \lambda, \tag{26}$$

which, when set to zero, leads to

$$q_k(f_k) \propto p_k(f_k) \exp\left( -\left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle (\gamma \ell^T M_i f - I_i)^2 \right\rangle_{q(\ell, f_{-k}, \gamma)} \right).$$

We can now expand the summand,

$$
\begin{aligned}
\left\langle(\gamma\ell^T M_i f - I_i)^2\right\rangle_{q(\ell,f_{-k})} &= \left\langle (\gamma \sum_m \sum_{s\neq k} \ell_m M_{i_{ms}} f_s - I_i + \gamma \sum_m \ell_m M_{i_{mk}} f_k)^2 \right\rangle_{q(\ell,f_{-k},\gamma)} \\
&= \left\langle \gamma^2 \right\rangle_{q(\gamma)} \left\langle \left( \sum_m \ell_m M_{i_{mk}} \right)^2 \right\rangle_{q(\ell)} f_k^2 \\
&+ 2 \left\langle \left( \sum_m \sum_{s\neq k} \gamma \ell_m M_{i_{ms}} f_s - I_i \right) \left( \sum_m \gamma \ell_m M_{i_{mk}} \right) \right\rangle_{q(\ell,f_{-k},\gamma)} f_k \\
&+ ... \\
&= B_{ki} f_k^2 + C_{ki} f_k + ...
\end{aligned}
$$

and substitute this into the expression for $q_k(f_k)$ to obtain

$$q_k(f_k) \propto p_k(f_k) \exp\left( - \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} (B_{ki} f_k^2 + C_{ki} f_k) \right). \qquad (27)$$

Substituting $p_k(f_k)$ into this expression, we finally arrive at

$$
\begin{aligned}
q_k(f_k) &\propto \exp(-\lambda_k f_k) \exp\left( - \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} (B_{ki} f_k^2 + C_{ki} f_k) \right) \\
&= \exp\left( \left( - \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} B_{ki} \right) f_k^2 + \left( \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} C_{ki} - \lambda_k \right) f_k \right)
\end{aligned}
$$

for $f_k \geq 0$ and $q_k(f_k) = 0$ otherwise. After completing the squares, this last equation leads to $q_k(f_k) \sim N_R(u_k, w_k)$, where $u_k, w_k$ are as defined in Eqs. 13-23.

### 3.2 Parametric form and update equations for $q(\gamma)$

Taking the variational derivative of $C_{KL}$ with respect to $q(\gamma)$ we obtain

$$
\begin{aligned}
\frac{dC_{KL}}{dq(\gamma)} &= \log q(\gamma) - \log p(\gamma) && (28) \\
&+ \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle (\gamma \ell^T M_i f - I_i)^2 \right\rangle_{q(\ell,f)} && (29) \\
&+ 1 + \lambda, && (30) \\
& && (31)
\end{aligned}
$$

which, when set to zero, leads to

$$q(\gamma) \propto p(\gamma) \exp\left( - \left\langle \frac{\sigma^{-2}}{2} \right\rangle_{q(\sigma^{-2})} \sum_{i=1}^{N} \left\langle (\gamma \ell^T M_i f - I_i)^2 \right\rangle_{q(\ell,f)} \right).$$

As before, we expand

$$\left\langle(\gamma\ell^T M_i f - I_i)^2\right\rangle_{q(\ell,f)} = \left\langle\left(\gamma\sum_m\sum_{s\neq k}\ell_m M_{i_{ms}} f_s - I_i + \gamma\sum_m \ell_m M_{i_{mk}} f_k\right)^2\right\rangle_{q(\ell,f_{-k},\gamma)}$$
$$= \left\langle(\ell^T M_i f)^2\right\rangle_{q(\ell,f)}\gamma^2 - 2I_i\left\langle\ell^T M_i f\right\rangle_{q(\ell,f)}\gamma + ...$$
$$= G_i\gamma^2 + H_i\gamma + ...$$

and substitute into the expression for $q(\gamma)$ to obtain

$$q(\gamma) \propto p(\gamma)\exp\left(-\left\langle\frac{\sigma^{-2}}{2}\right\rangle_{q(\sigma^{-2})}\sum_{i=1}^{N}\left(G_i\gamma^2 + H_i\gamma\right)\right). \quad (32)$$

Then, as before, we substitute the expression for $p(\gamma)$, rearrange terms and complete the squares to arrive at $q(\gamma) \sim N_R(u_\gamma, w_\gamma)$, where $u_\gamma, w_\gamma$ are as defined in Eqs. 13-23.

## 4 Cost function $C_{KL}$ in Algorithm 1

To derive the expression for the cost function $C_{KL}$ in Algorithm 1 of [3] we follow Miskin [1] and Miskin and MacKay [2] to approximate some of the terms in Eq. 11 of [3] with their upper bound. This allows us to re-write this expression as

$$C_{KL} = C_{KL}^{(\ell)} + C_{KL}^{(f)} + C_{KL}^{(\sigma^{-2})} + C_{KL}^{(\gamma)} - <\log p(D|\theta)>_{q(\theta)}, \quad (33)$$

with each term as given below.

$$C_{KL}^{(\ell)} = \sum_{m=2}^{M}\left\langle\log q_m(\ell_m)\right\rangle_{q_m(\ell_m)} - \left\langle\log p_m(\ell_m)\right\rangle_{q_m(\ell_m)} \quad (34)$$
$$\leq \sum_{m=2}^{M}\left(-\frac{1}{2}\log 2\pi w_m - \frac{1}{2w_m}\left\langle(\ell_m - u_m)^2\right\rangle_{q_m(\ell_m)}\right)$$
$$-\sum_{m=2}^{M}\sum_{n=1}^{N_m}\delta_{nm}\left(\log\frac{\pi_{nm}}{\delta_{nm}} + \left\langle\log p_{nm}(\ell_m)\right\rangle_{q_m(\ell_m)}\right)$$

$$C_{KL}^{(f)} = \sum_{k=1}^{K}\left\langle\log q(f_k)\right\rangle_{q_k(f_k)} - \left\langle\log p_k(f_k)\right\rangle_{q_k(f_k)} \quad (35)$$
$$= \sum_{k=1}^{K}\left(-\frac{1}{2}\log 2\pi w_k - \frac{1}{2w_k}\left\langle(f_k - u_k)^2\right\rangle_{q_k(f_k)}\right)$$
$$-\sum_{k=1}^{K}\left(\log\lambda_k - \lambda_k\left\langle f_k\right\rangle_{q_k(f_k)}\right)$$

$$C_{KL}^{(\sigma^{-2})} = \left\langle \log q(\sigma^{-2}) \right\rangle_{q(\sigma^{-2})} - \left\langle \log p(\sigma^{-2}) \right\rangle_{q(\sigma^{-2})} \tag{36}$$
$$= \log \frac{\Gamma(b)}{\Gamma(b_p)} + \log \frac{a_p^{b_p}}{a^b} + (b_p - b) \left\langle \log \sigma^{-2} \right\rangle_{q(\sigma^{-2})} + (a - a_p) \left\langle \sigma^{-2} \right\rangle_{q(\sigma^{-2})}$$

$$C_{KL}^{(\gamma)} = \langle \log q(\gamma) \rangle_{q(\gamma)} - \langle \log p(\gamma) \rangle_{q(\gamma)} \tag{37}$$
$$= -\frac{1}{2} \log 2\pi w_\gamma - \frac{1}{2w_\gamma} \left\langle (\gamma - u_\gamma)^2 \right\rangle_{q(\gamma)} - \log \lambda_\gamma + \lambda_\gamma \langle \gamma \rangle_{q(\gamma)}$$

$$\langle \log p(D|\theta) \rangle_{q(\theta)} = -\frac{N}{2} \log 2\pi + \frac{N}{2} \left\langle \log \sigma^{-2} \right\rangle_{q(\sigma^{-2})} - \tag{38}$$
$$- \frac{1}{2} \left\langle \sigma^{-2} \right\rangle \sum_{i=1}^{N} \left\langle (\ell^T M_i f - I_i)^2 \right\rangle_{q(\ell,f)}$$

## 5 Simplified expressions for Eqs. 13-23

$$\left\langle (\sum_m \ell_m M_{i_{mk}})^2 \right\rangle_{q(\ell)} = (\left\langle \ell^2 \right\rangle - \langle \ell \rangle)^T M_{i_{(:,k)}}^{(2)} + (\langle \ell \rangle^T M_{i_{(:,k)}})^2,$$

where the notation $X^{(2)}$ corresponds to the matrix or vector whose entries are the squares of the entries of $X$, $M_{i_{(:,k)}}$ refers to the $k^{th}$ column of matrix $M_i$, and $M_{i_{(k,:)}}$ refers to the $k^{th}$ row.

$$\left\langle \left( \sum_k M_{i_{mk}} f_k \right)^2 \right\rangle_{q(f)} = M_{i_{(:,k)}}^{(2)} (\left\langle f^2 \right\rangle - \langle f \rangle) + (M_{i_{(m,:)}} \langle f \rangle)^2.$$

$$\begin{aligned} \left\langle (\ell^T M_i f)^2 \right\rangle_{q(\ell,f)} &= (\langle \ell \rangle^T M_i \langle f \rangle)^2 + \langle f \rangle^T M_i^T \mathrm{cov}(\ell) M_i \langle f \rangle \\ &+ \langle \ell \rangle^T M_i \mathrm{cov}(f) M_i^T \langle \ell \rangle \\ &+ (\left\langle \ell^2 \right\rangle - \langle \ell \rangle^2)^T \mathrm{diagv}(M_i \mathrm{cov}(f) M_i^T), \end{aligned}$$

where $\mathrm{diagv}(X)$ corresponds to a vector whose entries are the diagonal entries of matrix $X$.

$$\frac{C_{ki}}{2} = -I_i \langle \gamma \rangle \langle \ell \rangle^T M_{i_{(:,k)}} + \left\langle \gamma^2 \right\rangle \langle f \rangle_{-k}^T M_{i_{(:,-k)}}^T (\mathrm{cov}(\ell) + \langle \ell \rangle \left\langle \ell^T \right\rangle) M_{i_{(:,k)}},$$

where the notation $X_{-k}$ corresponds to the vector $X$ with its $k^{th}$ entry removed, $M_{i_{(:,-k)}}$ refers to the matrix formed by removing the $k^{th}$ column of $M_i$, and $M_{i_{(-k,:)}}$ is the matrix formed by removing its $k^{th}$ row.

$$\frac{E_{mi}}{2} = -I_i \langle \gamma \rangle M_{i_{(m,:)}} \langle f \rangle + \left\langle \gamma^2 \right\rangle \langle \ell \rangle_{-m}^T M_{i_{(-m,:)}} (\mathrm{cov}(f) + \langle f \rangle \langle f \rangle^T) M_{i_{(m,:)}}^T$$

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