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# Toward Shape from a Single Specular Flow

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## Abstract

In “Shape From Specular Flow: Is One Flow Enough?” (Vasilyev, et al., 2011 [5]) we show that mirror shape can often be reconstructed from the observation of a single specular flow. In this technical report we provide additional details that, due to space constraints, could not be included in the paper. First we provide a derivation of the linear system for the reflection field derivative in the direction orthogonal to the flow,  $\hat{\mathbf{r}}_y$ . Second, we derive an expression for the determinant of this system which is independent of coordinate system. Third, we show that the sphere is reconstructable whenever the scene rotation is neither on the equator nor parallel to the view direction. Finally we provide additional details for the outline of the proof that reconstructability is a generic property and for our numerical investigation of the dimensionality of the variety described by the “bad” conditions.

## 1 Linear system for $\hat{\mathbf{r}}_y$

### 1.1 Derivation of equation (6) in [5]

Let  $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = (\hat{u}_1, \hat{u}_2)^\top$ . Starting with equation (5) in [5],

$$\nabla_y (\nabla_x \hat{\mathbf{r}}) = \nabla_y (\mathbf{J}x) = \nabla_y (\hat{\mathbf{r}}_x \hat{u}_1 + \hat{\mathbf{r}}_y \hat{u}_2) = \nabla_y \left( \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_{\times} \hat{\mathbf{r}} \right) \quad (1)$$

and expanding using product rule we get:

$$\nabla_y (\hat{\mathbf{r}}_x \hat{u}_1 + \hat{\mathbf{r}}_y \hat{u}_2) = \nabla_y (\hat{\mathbf{r}}_x) \hat{u}_1 + \hat{\mathbf{r}}_x \nabla_y (\hat{u}_1) + \nabla_y (\hat{\mathbf{r}}_y) \hat{u}_2 + \hat{\mathbf{r}}_y \nabla_y (\hat{u}_2) \quad (2)$$

and

$$\nabla_y \left( \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_{\times} \hat{\mathbf{r}} \right) = \left( \frac{1}{\|\mathbf{u}\|} \right)_y [\boldsymbol{\omega}]_{\times} \hat{\mathbf{r}} + \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_{\times} \hat{\mathbf{r}}_y \quad (3)$$

Now, in the flow coordinate system, at the point  $\mathbf{x}$  we have  $\hat{u}_1 = 1, \hat{u}_2 = 0, \nabla_y(\hat{u}_1) = 0$ , and  $\nabla_y(\hat{u}_2) = \kappa_\perp$ , so we have

$$\nabla_y(\hat{\mathbf{r}}_x) + \hat{\mathbf{r}}_y \kappa_\perp = \left( \frac{1}{\|\mathbf{u}\|} \right)_y [\boldsymbol{\omega}]_\times \hat{\mathbf{r}} + \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_\times \hat{\mathbf{r}}_y. \quad (4)$$

Equality of mixed partials of  $\hat{\mathbf{r}}$  at  $\mathbf{x}$  means that

$$\hat{\mathbf{r}}_{yx} = \left( \frac{1}{\|\mathbf{u}\|} \right)_y [\boldsymbol{\omega}]_\times \hat{\mathbf{r}} + \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_\times \hat{\mathbf{r}}_y - \hat{\mathbf{r}}_y \kappa_\perp \quad (5)$$

which is equation (6) in [5].

## 1.2 Derivation of equation (7) in [5]

Differentiating the the integrability constraint in equation (4) of [5] in the direction of the flow at  $\mathbf{x}$ , that is the  $x$  direction, we get:

$$\begin{pmatrix} -(r_3 + 1) \\ 0 \\ r_1 \end{pmatrix}^\top \hat{\mathbf{r}}_{yx} + \begin{pmatrix} -(r_3 + 1) \\ 0 \\ r_1 \end{pmatrix}^\top_x \hat{\mathbf{r}}_y = \begin{pmatrix} 0 \\ -(r_3 + 1) \\ r_2 \end{pmatrix}^\top_x \hat{\mathbf{r}}_x + \begin{pmatrix} 0 \\ -(r_3 + 1) \\ r_2 \end{pmatrix}^\top \hat{\mathbf{r}}_{xx} \quad (6)$$

Now differentiating the specular flow derivative equation (1) in [5] in the flow direction  $x$  we have

$$\nabla_x(\hat{\mathbf{r}}_x \hat{u}_1 + \hat{\mathbf{r}}_y \hat{u}_2) = \nabla_x(\hat{\mathbf{r}}_x) \hat{u}_1 + \hat{\mathbf{r}}_x \nabla_x(\hat{u}_1) + \nabla_x(\hat{\mathbf{r}}_y) \hat{u}_2 + \hat{\mathbf{r}}_y \nabla_x(\hat{u}_2) = \nabla_x \left( \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_\times \hat{\mathbf{r}} \right) \quad (7)$$

and since in the flow coordinate system, at the point  $\mathbf{x}$  we have  $\hat{u}_1 = 1, \hat{u}_2 = 0, \nabla_x(\hat{u}_1) = 0$ , and  $\nabla_x(\hat{u}_2) = \kappa$ ,

$$\hat{\mathbf{r}}_{xx} = \nabla_x \left( \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_\times \hat{\mathbf{r}} \right) - \hat{\mathbf{r}}_y \kappa. \quad (8)$$

Note that the first term on the right hand side of equation 6 is zero. Now, substituting the expression for  $\hat{\mathbf{r}}_{xx}$  into equation 6 we get

$$\begin{pmatrix} -(r_3 + 1) \\ 0 \\ r_1 \end{pmatrix}^\top \hat{\mathbf{r}}_{yx} + \begin{pmatrix} -r_{3x} \\ -(r_3 + 1)\kappa \\ r_{1x} + r_2\kappa \end{pmatrix}^\top_x \hat{\mathbf{r}}_y = \begin{pmatrix} 0 \\ -(r_3 + 1) \\ r_2 \end{pmatrix}^\top_x \nabla_x \left( \frac{1}{\|\mathbf{u}\|} [\boldsymbol{\omega}]_\times \hat{\mathbf{r}} \right) \triangleq b(\hat{\mathbf{r}}, \hat{\mathbf{r}}_x) \quad (9)$$

where the right hand side can be written as a scalar function at  $\mathbf{x}$ :

$$\begin{aligned} b &= r_2 \left( \left( \frac{1}{\|\mathbf{u}\|} \right)_x (-\omega_2 r_1 + \omega_1 r_2) - \frac{1}{\|\mathbf{u}\|} (-\omega_2 r_{1x} + \omega_1 r_{2x}) \right) + \\ &\quad -(r_3 + 1) \left( \left( \frac{1}{\|\mathbf{u}\|} \right)_x (\omega_3 r_1 - \omega_1 r_3) + \frac{1}{\|\mathbf{u}\|} (\omega_3 r_{1x} - \omega_1 r_{3x}) \right). \end{aligned}$$

## 2 Determinant of the Linear System for $\hat{\mathbf{r}}_y$

### 2.1 Derivation of equation (9) in [5]

The determinant of the matrix in equation (8) of [5] is

$$b_1 r_1 r_2 + b_2 (-r_1^2 - r_3^2 - r_3) + b_3 (r_2 (r_3 + 1)) \quad (10)$$

Substituting the expressions for  $b_1, b_2, b_3$  from the paper, replacing  $r_1^2 + r_3^2$  with  $1 - r_2^2$  and canceling terms yields

$$r_2 ((r_3 + 1) r_{1x} - r_1 r_{3x}) + (r_3 + 1)^2 \kappa + \frac{1}{\|\mathbf{u}\|} (\omega_1 r_1 (r_2^2 - (r_3 + 1)) + \omega_2 r_2 (-(r_3 + 1)^2 - r_1^2) + \omega_3 (r_3 + 1) (r_2^2 - (r_3 + 1))) \quad (11)$$

Now using the fact that  $r_1^2 + r_2^2 + r_3^2 = 1$  and the specular flow derivative equation (1) in [5] to substitute for  $r_{1x}$  and  $r_{3x}$ , we get:

$$\begin{aligned} & r_2 ((r_3 + 1) \left( \frac{-\omega_3}{\|\mathbf{u}\|} r_2 + \frac{\omega_2}{\|\mathbf{u}\|} r_3 \right) - r_1 \left( \frac{-\omega_2}{\|\mathbf{u}\|} r_1 + \frac{\omega_1}{\|\mathbf{u}\|} r_2 \right)) + (r_3 + 1)^2 \kappa \\ & + \frac{1}{\|\mathbf{u}\|} ((r_2^2 - (r_3 + 1)) (\omega_1 r_1 + \omega_3 (r_3 + 1)) + r_2 (r_2^2 - 2(r_3 + 1)) \omega_2). \end{aligned}$$

Simplifying we obtain:

$$(r_3 + 1) ((r_3 + 1) \kappa + \frac{\omega_1}{\|\mathbf{u}\|} (-r_1) + \frac{\omega_2}{\|\mathbf{u}\|} (-r_2) + \frac{\omega_3}{\|\mathbf{u}\|} (-(r_3 + 1))). \quad (12)$$

This last expression for the determinant is zero at those image points  $\mathbf{x}$  where

$$(r_3 + 1) = 0 \quad \text{or} \quad \kappa = \frac{\boldsymbol{\omega}^T \hat{\mathbf{r}} + \omega_3}{(r_3 + 1) \|\mathbf{u}\|}. \quad (13)$$

Note that  $(r_3 + 1) = 0$  only holds at points on the occluding contour and the latter expression is equation (9) in [5], and is the expression we wanted to derive.

### 2.2 Degenerate Flow Curves on the Unit Sphere

In this section we prove that in the case of the sphere only those scene rotation axes that lie on the equator or are parallel to the north and south pole result in degenerate flow curves.

We consider a global  $xyz$  coordinate system with the origin at the center of the sphere and the  $xy$  plane parallel to the image plane. We focus our attention on the visible hemisphere of the unit sphere which we parameterize over the  $xy$  plane as  $\hat{M}(x, y) = (x, y, f(x, y)) = (x, y, \sqrt{1 - x^2 - y^2})$ . Because of the symmetry of the sphere we do not need to look at every possible scene rotation  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ , and consider only those that lie in the  $xz$  plane, that is,  $\boldsymbol{\omega} = (w_1, 0, w_3)$ .

Let  $D$  be the set of degenerate points. This set is the union of the set of points where the condition in equation (12) of the paper holds with the silhouette ( $f = 0$ ):

$$D = \left\{ (x, y) : \left( \kappa \|\mathbf{u}\| - \frac{\boldsymbol{\omega}^T \hat{\mathbf{r}} + \omega_3}{(r_3 + 1)} \right) f = 0 \right\}. \quad (14)$$

The flow curves on the sphere can also be written as level sets parameterized by  $C \in [-\|\boldsymbol{\omega}\|, \|\boldsymbol{\omega}\|]$  :

$$V_C = \{(x, y) : \boldsymbol{\omega}^\top \hat{\mathbf{r}} - C = 0\}. \quad (15)$$

Our goal is to determine the values of  $\omega_1, \omega_3$ , and  $C$  for which the set  $V_C$  is a subset of the set  $D$ .

We begin by expressing the conditions that define these sets,  $D$  and  $V_C$ , in terms of the parameters  $x, y$  and the height function  $f$ .

The unit normal and unit reflection vector at the point  $(x, y)$  are related by  $\hat{\mathbf{r}} = 2\hat{\mathbf{n}}^\top \hat{\mathbf{v}}\hat{\mathbf{n}} - \hat{\mathbf{v}}$  where  $\hat{\mathbf{v}}$  is the view direction which we take to be in the direction of the positive  $z$  axis. For the sphere, the unit normal in terms of  $x$  and  $y$  is  $\hat{\mathbf{n}} = (x, y, f)^\top$ , and so the unit reflection vector field is

$$\hat{\mathbf{r}} = (2xf, 2yf, 2f^2 - 1)^\top \quad (16)$$

Using the SFSF equation (1) in [5] and the derivatives of  $\hat{\mathbf{r}}$  we can express the flow  $\mathbf{u} = (u_1, u_2)$  as

$$\begin{aligned} u_1 &= \frac{-r_2(\omega_3 r_{3y} + w_1 r_{1y})}{r_{1x} r_{3y} - r_{3x} r_{1y}} = - \left( w_3 y + \frac{w_1 x y}{2f} \right) \\ u_2 &= \frac{r_2(\omega_3 r_{3x} + w_1 r_{1x})}{r_{1x} r_{3y} - r_{3x} r_{1y}} = \left( w_3 x + \frac{w_1 (f^2 - x^2)}{2f} \right) \end{aligned} \quad (17)$$

The product  $\kappa\|\mathbf{u}\|$ , in equation (9) in [5] has the following expression in terms of the flow components and their derivatives

$$\kappa\|\mathbf{u}\| = \frac{u_1^2 u_{2x} - u_2^2 u_{1y} + u_1 u_2 (u_{2y} - u_{1x})}{u_1^2 + u_2^2} \quad (18)$$

Substituting the expressions in equation (18), the set  $D$  can be defined as follows:

$$D = \{(x, y) : g(x, y) = f(xf^4 - 2x^3f^2 + x^3y^2 + x^5 - 3xy^2f^2)\omega_1^3 + \quad (19)$$

$$f(-4x^2f^3 + 4x^4f + 4x^2y^2f - 4y^2f^3)\omega_1^2\omega_3 + \quad (20)$$

$$f(4x^3f^2 + 4xy^2f^2)\omega_3^2\omega_1 = 0\} \quad (21)$$

Note that when  $\omega_1 = 0$  this equation is trivially satisfied – that is in this case every image point  $(x, y)$  is degenerate.

To express  $V_C$  in terms of  $x, y, f$ , we note that each flow curve on the sphere satisfies  $\boldsymbol{\omega}^\top \hat{\mathbf{r}} = C$  for  $C \in [-\|\boldsymbol{\omega}\|, \|\boldsymbol{\omega}\|]$ . Thus each flow curve is a zero level set

$$V_C = \left\{ (x, y) : h_C(x, y) = \omega_1 x f + \omega_3 f^2 - \frac{C + \omega_3}{2} = 0 \right\}. \quad (22)$$

Our goal now is to determine the values of  $\omega_1, \omega_3$ , and  $C$  for which  $V_C \not\subseteq D$ . For such values of  $\omega_1, \omega_3$ , and  $C$  the flow curve will not be degenerate.

In general the question of whether the zero level set of a real valued function  $f_1$ ,  $A = \{(x, y) : f_1(x, y) = 0\}$ , is a subset of the zero level set of another real valued function  $f_2$ ,  $B = \{(x, y) : f_2(x, y) = 0\}$ , is difficult to answer. However when the functions  $f_1$  and  $f_2$  are polynomials, this can be done by polynomial division. In that case  $A \subseteq B$  if and only if the remainder of polynomial division of  $f_2$  by  $f_1$  is the zero polynomial (see for example page 82 corollary 2 in [2]).

Note that we can write both  $g$  and  $h_C$  as  $g = g_e + g_o$  and  $h_C = h_{C_e} + h_{C_o}$  where  $g_e, h_{C_e}$  involve all and only those terms of  $g$  and  $h_C$  (respectively) that have even powers or  $f$ . Similarly  $g_o, h_{C_o}$  involve all and only those terms that have the odd powers of  $f$ . Using these expressions we form the following sets

$$\tilde{D} = \{(x, y) : G(x, y) = (g_e + g_o)(g_e - g_o) = g_e^2 - g_o^2 = 0\} \quad (23)$$

$$\tilde{V}_C = \{(x, y) : H(x, y) = (h_{C_e} + h_{C_o})(h_{C_e} - h_{C_o}) = h_{C_e}^2 - h_{C_o}^2 = 0\} \quad (24)$$

where again  $C \in [-\|w\|, \|w\|]$

Note that both  $G$  and  $H$  are polynomials in  $x, y$  and that  $D \subseteq \tilde{D}$ ,  $V_C \subseteq \tilde{V}_C$ . Additionally we now prove that  $V_C \subseteq D$  implies  $\tilde{V}_C \subseteq \tilde{D}$ . First note that  $V_C \subseteq D$  and  $D \subseteq \tilde{D}$  imply that  $V_C \subseteq \tilde{D}$ . The set  $D^* = \{(x, y) : (g_e - g_o) = 0\} \subseteq \tilde{D}$  and hence the set  $V_C \cup D^* = \{(x, y) : (h_{C_e} + h_{C_o})(g_e - g_o) = 0\} = \{(x, y) : (h_{C_e} - h_{C_o})(g_e + g_o) = 0\} \subseteq \tilde{D}$ . This means that the set  $V_C^* = \{(x, y) : (h_{C_e} - h_{C_o}) = 0\} \subseteq \tilde{D}$ . Finally since  $V_C^* \subseteq \tilde{D}$  and  $D \subseteq \tilde{D}$ , their union,  $V_C^* \cup D = \tilde{V}_C \subseteq \tilde{D}$ .

Because  $G$  and  $H$  are polynomials, we know that  $\tilde{V}_C \subseteq \tilde{D}$  if and only if the remainder of the division of  $H$  by  $G$  is the zero polynomial. Hence if the remainder of the division of  $H$  by  $G$  is not the zero polynomial then  $\tilde{V}_C \not\subseteq \tilde{D}$  and by the above,  $V_C \not\subseteq D$  – that is the flow curve  $V_C$  is not degenerate.

Now the remainder,  $R$  of the division of  $H$  by  $G$  is a polynomial in  $xy$  with coefficients  $C_i, i \in \{1, \dots, 9\}$ . These coefficients are rational functions in  $\omega_1, \omega_3$ , and  $C$ . To simplify our analysis of the coefficients we add the following restriction on the values of  $\omega_1$  and  $\omega_3$ :  $\omega_3 = 1 - \omega_1^2$  for  $\omega_1 \in (0, 1]$  (recall that we already dealt with the case  $\omega_1 = 0$  above. This simplification is not very restrictive as we are really only concerned with the direction of  $\omega$  since changing the magnitude of  $\omega$  does not change the flow curves on the image plane. Using the substitution  $\omega_3 = 1 - \omega_1^2$ , the denominator of the rational functions  $C_i$  is

$$4(-7\omega_1^6 + 6\omega_1^4 + 6\omega_1^8 - 3\omega_1^2 - 3\omega_1^{10} + 1 + \omega_1^{12}) \quad (25)$$

This polynomial has no roots in  $\mathbb{R}$ , and so we can focus on analyzing just the numerators  $N_i$  of the  $C_i, i \in \{1, \dots, 9\}$ .

Let  $\mathbb{C}[\omega_1, C]$  be the set of complex polynomials in two variables. Then considering the  $N_i$  as elements of  $\mathbb{C}[\omega_1, C]$ , let the set  $\langle N_1, \dots, N_9 \rangle$  be the ideal generated by the  $N_i$ . (see for example [2]). The remainder  $R$  is the zero polynomial if and only if all of the  $N_i$  are simultaneously zero. This happens when  $(\omega_1, C)$  are elements of the affine variety

$$V(\langle N_1, \dots, N_9 \rangle) = \{(\omega_1, C) \in \mathbb{C} : N_i(\omega_1, C) = 0, i = 1, \dots, 9\}, \quad (26)$$

that is when  $(\omega_1, C)$  is a root of each  $N_i$ . It is a well known fact that if the ideal generated by the  $N_i$ ,  $\langle N_1, \dots, N_9 \rangle$ , is equal to the ideal generated by some polynomials  $M_1, \dots, M_s \in \mathbb{C}[\omega_1, C]$ ,  $\langle M_1, \dots, M_s \rangle$ , then  $V(\langle N_1, \dots, N_9 \rangle) = V(\langle M_1, \dots, M_s \rangle) = \{(\omega_1, C) \in \mathbb{C} : M_j(\omega_1, C) = 0, j = 1, \dots, s\}$ .

Thus we can analyze the roots of polynomials  $M_1, \dots, M_s$  to determine when  $R$  is the zero polynomial. One such set of  $M_1, \dots, M_s$  is given by the Groebner basis of the ideal  $\langle N_1, \dots, N_9 \rangle$ :

$$\{ C^2(C^3 + C^2 - C - 1), \quad (27)$$

$$C(\omega_1^2 C + \omega_1^2 + C^3 + C^2 - C - 1), \quad (28)$$

$$\omega_1^2(\omega_1^2 - 2) - 2\omega_1^2 C + C(-C^3 - 2C^2 + 2) + 1\} \quad (29)$$

The only values of  $(\omega_1, C)$  in  $\mathbb{R}^2$  that are the roots of these equations simultaneously are  $(1, 0)$ ,  $(-1, 0)$ , and  $(0, -1)$ . This means that there are no degenerate flow curves when the scene rotation does not lie on the equator and is not parallel to the view axis.

Note that because  $\tilde{V}_C \subseteq \tilde{D}$  does not imply  $V_C \subseteq D$ , the above analysis does not necessarily imply that for the scene rotation that lies on the equator,  $\omega_1 = 1, \omega_3 = 0$ , the flow curve  $\omega^\top \hat{\mathbf{r}} = C = 0$  is a degenerate flow curve. We can show that this is indeed true by inspecting the sets  $V_C$  and  $D$ .  $V_C = \{(x, y) : xf = 0\}$  and  $D = \{(x, y) : xf(f^4 - 2x^2f^2 + x^2y^2 + x^4 - 3y^2f^2) = 0\}$  when  $\omega_1 = 1, \omega_3 = 0$ , and  $C = 0$ , so, clearly,  $V_C \subseteq D$  – this flow curve is degenerate.

To summarize, for the sphere, the only cases in which there is a degenerate flow curve are

- $\omega_1 = 0$
- $\omega_3 = 0, C = 0$

### 3 Reconstruction in the General Case

In section 4.3 of [5] we discuss the plausibility of reconstructing the geometry of general surfaces observed under a single known scene rotation. In the next few section we present a more rigorous analysis. As mentioned in the paper, our analysis parallels that of generic properties of curves presented in [1].

#### 3.1 Transversality

In this section we explain why we do not expect the image of the Monge-Taylor map of order  $n$  of a generic surface to intersect any algebraic subset of  $\mathbb{R}^n$  with co-dimension three or higher. This is a basic application of Thom’s transversality lemma (see for example [1]) and our argument closely parallels that in [1], where the lemma is used to prove that certain properties of curves are generic. We will need a few changes, however, to apply this theorem properly in our setting.

In order to apply known results, we would like to work with surface points over a closed connected subset  $K$  of  $U$ . We argue below that for a generic set of height fields over the set  $K$ , the property that their Monge-Taylor maps do not intersect a fixed algebraic subset of co-dimension three or greater is generic. We begin by defining what we mean by a generic property of smooth height fields.

**Definition 1.** A property  $P$  is said to be generic or to hold for a generic set of smooth height fields if it is open and dense.

where dense and open are defined as follows:

**Definition 2.** A property  $P$  is said to be dense or to hold for a dense set of smooth height fields  $M : K \rightarrow \mathbb{R}^3$  if the following holds. For any such  $M$  there should be an open neighborhood  $W$  of 0 in  $\mathbb{R}^n$  and a smooth family of height fields  $\hat{M} : K \times W \rightarrow \mathbb{R}^3$  such that (i)  $\hat{M}(\cdot, 0) = M(\cdot)$  and (ii) for some sequence  $\{w_n\}$  in  $W$  with  $\lim_{n \rightarrow \infty} w_n = 0$  property  $P$  holds for the sequence of height fields  $M_n$  defined by  $M_n = \hat{M}(\cdot, w_n)$ .

**Definition 3.** A property  $P$  is said to be open or to hold for an open set of smooth height fields if given such a height field  $M$  with property  $P$  and any smooth family of height fields  $\hat{M} : K \times W \rightarrow \mathbb{R}^3$ ,  $\hat{M}_w(\cdot) = \hat{M}(\cdot, w)$ , the property  $P$  holds for all height fields  $\hat{M}_w$  for  $w$  in some neighborhood  $W_0$  of 0.

We now describe the Monge-Taylor map of order  $k$ , which allows us to represent the surface as a subset of  $\mathbb{R}^n$ , for some positive  $n$ . The Taylor approximations of order  $k$  at each surface point are polynomials which carry all the infinitesimal information of the surface up to order  $k$ . Each such polynomial is uniquely determined by  $n$  numbers, the coefficients of these polynomials,  $(a_1, \dots, a_n)$ , allowing us to identify the space  $V_n$  of polynomials in two variables of degree  $\leq n$  with  $\mathbb{R}^n$ . The Monge-Taylor map of order  $k$  is a map from the domain  $K$  to  $V_n$  which associates to each  $\mathbf{x} \in K$  the Taylor approximation of order  $k$  of the surface at  $\mathbf{x}$ . Its image is a set  $\{(a_1(\mathbf{x}), \dots, a_n(\mathbf{x})) : \mathbf{x} \in K\} \subset \mathbb{R}^n$ . Thus, we represent the surface as a subset of  $\mathbb{R}^n$  using the Taylor coefficients of the local  $k$ -th order Taylor approximations to the surface. One way to obtain this map is to write the surface at each  $(x, y) \in K$  as the graph of a function  $f_{xy}(o, u)$  and then take the Taylor expansion of this function (note that the subscript  $xy$  refers to the point about which  $f$  is expanded rather than differentiation). In our case this is easy to do since the surface is assumed to be a graph of a function. We simply move the domain orthogonally to the surface until it passes through  $f(x, y)$  and then set the point  $(x, y)$  to be the origin. In this way  $f_{xy}(0, 0) = 0$  and the position term does not appear in the Taylor approximation. (In fact in our case  $f_{xy}$  can be defined in terms of the height field  $f$  of the surface as  $f_{xy}(o, u) = f(o - x, u - y) - f(x, y)$ ). Based on this discussion we define the Monge-Taylor map as follows:

**Definition 4.** The Monge-Taylor map of order  $k$  is a map  $m_M : K \rightarrow \mathbb{R}^n$  which associates to each  $(x, y) \in K$  the  $k$ -jet of the function  $f_{xy}$  at  $(0, 0)$  truncated to degree  $k$ . That is  $m_M(x, y) = j^k f_{xy}(0, 0)$ .

Using the above definitions, we can state the main theorem as follows:

**Theorem 1.** *A generic set of height fields have the property that their Monge-Taylor maps do not intersect a fixed algebraic subset  $B$  of  $\mathbb{R}^n$  of co-dimension three or greater.*

**Remark 1.** This theorem and its proof are essentially those of Corollary 9.7 of [1]. The main difference is that the set  $B$  is an algebraic subset instead of a closed manifold. By note (ii) of Trotman [4], Thom's transversality lemma applies to this case as well, which shows that transversality of mappings to  $B$  is a dense property. To prove openness, instead of directly applying Proposition 8.23 of [1], we use the theorem in [4] due to Trotman that shows that transverse intersection to a fixed Whitney stratification is an open property of maps. This fact, together with Whitney's theorem [6] that every algebraic subset admits a Whitney stratification, proves that transversality of mappings to  $B$  is an open property.

A minor difference from [1] is that the domain  $K$  in our case is a compact manifold with boundary. The transversality arguments can be extended to manifolds with boundary as shown in chapter 3 of Hirsch [3].

In the next section we form an algebraic set  $B$  such that Theorem 1 implies that strong reconstructability is a property of a generic set of height fields.

## 4 The set $B$

In our context we describe an algebraic subset  $B$  of  $\mathbb{R}^n$  such that if the Monge-Taylor map of our height field misses  $B$ , we will know that our reconstruction method will succeed over the entire surface. Because all conditions for strong reconstructability can be written in terms of the scene rotation  $\omega$  and the derivatives of the height function  $f(\mathbf{x})$ , we can describe the set of points at which strong reconstructability does *not* hold using the surface representation obtained via the Monge-Taylor map. For a

given rotation  $\omega$  a surface does not have the strong reconstructability property if there is a degenerate point at which the flow curve through that point is simultaneously tangent to and has the same curvature as a degenerate curve. For simplicity we call such a point a *bad point*. To describe the set of bad points we do the following. First we write down the conditions for bad points in terms of  $\omega$  and derivatives of the height function  $f(\mathbf{x})$  only. Then we identify these derivatives with the coefficients of the  $k$ -th order Taylor map at each point, obtaining a representation of these conditions in the same space in which the image of the Monge-Taylor map represents the surface.

At a bad point the following three conditions, called *bad conditions* must be satisfied.

1. The point must be degenerate. That is equation (9) in the paper must hold:

$$D(a_1, \dots, a_n) = \kappa \|\mathbf{u}\| (r_3 + 1) - \omega^T \hat{\mathbf{r}} - \omega_3 = 0, \quad (30)$$

where we obtain the expression in terms of the Taylor coefficients  $a_i$  at  $\mathbf{x}$  by writing out the middle expression in terms of the derivatives of  $f$  evaluated at  $\mathbf{x}$  and identifying them with the  $a_i(\mathbf{x})$ .

2. The flow curve and the degenerate curve must be tangent. That is if we take the gradient (derivatives with respect to  $\mathbf{x}$ ) of  $D = 0$ , it must be perpendicular to the flow. Again note that our procedure is to take derivatives of equation (30) with respect to components of  $\mathbf{x}$ , obtaining a condition in terms of  $\omega$  and derivatives of the height function  $f(\mathbf{x})$ , and then to identify the derivatives of  $f(\mathbf{x})$  with the  $a_i(\mathbf{x})$ . Hence we have

$$T(a_1, \dots, a_n) = D_x u_1 + D_y u_2 = 0, \quad (31)$$

where subscripts of  $D$  denote derivatives and  $\hat{u}_1, \hat{u}_2$  are the components of the normalized flow.

3. The flow curve and degenerate curve must have equal curvatures:

$$C(a_1, \dots, a_n) = \kappa_n (D_x^2 + D_y^2)^{3/2} - (D_y^2 D_{xx} - 2D_x D_y + D_x^2 D_{yy}) \kappa_d = 0. \quad (32)$$

where  $\kappa_n$  and  $\kappa_d$  are the numerator and denominator of the expression for the flow curve curvature:

$$\kappa = \frac{\kappa_n}{\kappa_d} = \frac{u_1^2 u_{2x} - u_2^2 u_{1y} + u_1 u_2 (u_{2y} - u_{1x})}{(u_1^2 + u_2^2)^{3/2}}$$

and  $u_{1x}, u_{1y}, u_{2x}, u_{2y}$  denote the derivatives of the flow components (again written in terms of the derivatives of  $f(\mathbf{x})$ ).

There are a few things to note about these conditions.

- If a degenerate point lies in a region of degenerate points rather than on a curve the above three conditions are still satisfied (as they should be, since these points are “bad” in the sense that the flow curve near such a point is clearly composed of degenerate points and reflection field reconstruction near this point could fail).
- Conditions  $D$  and  $T$  are polynomials in  $(a_1, \dots, a_n)$ . Condition  $C$  could be made a polynomial by moving the second product to the right hand side, squaring, and moving the result back. Note that the set of points  $(a_1, \dots, a_n)$  satisfying condition  $C$  and this modified version is exactly the same; therefore from this point on we will use  $C$  to refer to the modified version of this condition. Because the three conditions are polynomials in  $(a_1, \dots, a_n)$ , the points that satisfy them define an algebraic variety,  $B$  in  $\mathbb{R}^n$

- Since expressing the conditions  $D$ ,  $T$ , and  $C$  only requires derivatives of  $f(\mathbf{x})$  up to fifth order, a fifth order Taylor expansion suffices, which implies  $n = 20$ .

## 4.1 Dimension of $B$

If the co-dimension  $B$  is three or greater, we will be in good shape since Theorem 1 then implies that strong reconstructability is a generic property of height fields. This co-dimension is quite plausible due to the fact that it is cutout of  $\mathbb{R}^n$  using three polynomial equations. However, this is not sufficient to give us its dimension. Symbolic computations could be used to compute the dimension of  $B$ , but we have not been able to successfully run these algorithms on our polynomials (due to their large size). We have, however, performed some numerical experiments whose results are consistent with a co-dimension of three. The facts used in this discussion can be found in any standard text on Algebraic Geometry; see for example [2].

Let  $I(V)$  be the ideal of polynomials that vanish on an irreducible algebraic variety  $V \subset \mathbb{R}^n$ . We can always find a finite basis for this ideal, that is, a set of  $q$  polynomials  $p_1, \dots, p_q$  that generate this ideal. The Jacobian matrix  $J$  of this basis is the  $q \times n$  matrix with entries  $\frac{\partial p_k}{\partial a_i}$ . If we take any set of polynomials whose zero set is  $V$ , then the rank of  $J'$ , the Jacobian of this set of polynomials, can be smaller, but never larger than the rank of the Jacobian  $J$  of the basis. If  $r$  is the maximal rank achieved by the Jacobian  $J$  over points in  $V$ , then the dimension of  $V$  is  $n - r$ . Summing this up, the relevant conclusion is

**Theorem 2.** *Let  $x$  be a point in an irreducible algebraic variety  $V$ , and let  $r'(x)$  be the rank of any  $J'$  at  $x$ . Then the codimension of  $V$  is  $r'$  or greater.*

**Remark 2.** The difficulty in our case is that the algebraic variety  $B$  may not be irreducible. Say it can be written as a union of irreducible algebraic varieties  $B_j$ . In this case the above theorem would be limited because the codimension of one component of  $B$  does not inform us anything about the dimensions of the other components. Therefore, while we can numerically discover the dimension of some components of  $B$ , it is possible that we may be missing others. As such, this theorem can give us consistency, but it cannot give us proof.

In our case,  $B$  is a subset of  $\mathbb{R}^{20}$  generated by three polynomials  $D$ ,  $T$ ,  $C$  described above, though they might not form a basis for the ideal  $I(B)$ . In our experiments we find a point on  $B$  and determine the Jacobian of the three polynomials at this point. In all cases we find that the rank is three. Thus the maximal rank is at least three and the dimension at most seventeen. These results are encouraging. However, since  $B$  may be reducible, the correct conclusion to make from these experiments is that it has at least one component (a subvariety) of dimension seventeen. It could, as remarked above, have other components of different dimension. For this reason, these results are encouraging but not conclusive, and further analysis of the dimensionality of  $B$  is necessary.

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